

PALINDROMIC WIDTHS OF GRAPHS OF GROUPS

KRISHNENDU GONGOPADHYAY AND SWATHI KRISHNA

ABSTRACT. In this paper, we prove that the palindromic width of the fundamental group of a graph of groups is infinite. In particular, we prove infiniteness of the palindromic widths for HNN extensions of a group by proper associated subgroups and the amalgamated free products of two groups (except when the amalgamated subgroup has index two in each of the factors), thus answering Question 3 and Question 4 in *Comm Algebra*, vol 43, issue 11 (2015), 4809–4824.

1. INTRODUCTION

Words are basic objects in group theory and they are natural sources to view groups as geometric objects. Using words, one can naturally associated lengths to elements in a group and the maximum of all such lengths gives the notion of a width. The theory of verbal subgroup, that is subgroup determined by a word (for example, the commutator subgroup), and the verbal width have seen many decisive results in recent time. For example, the problem to determine the commutator width in finite simple groups has seen considerable attention in the literature, and has been resolved recently, see [LOST10]. For an exposition of verbal widths in groups, see the text [Seg09].

Influenced by the developments on verbal subgroups, it is natural to ask verbal widths given by other classes of words. In this paper, we consider width that comes from palindromic words. Let G be a group and let S be a generating set with $S^{-1} = S$. A *word-palindrome* or simply, *palindrome* in G is a reduced word in S which reads the same forward and backward. For an element $g \in G$, the *palindromic length*, $l_P(g)$ is the minimum number k such that g can be expressed as a product of k palindromes. Then the *palindromic width* of G with respect to S is defined as:

$$pw(G, S) = \sup_{g \in G} l_P(g).$$

Motivated by problems in model theory, Bardakov, Shpilrain and Tolstykh [BST05] initiated the investigation of palindromic width and proved that the palindromic width of a non-abelian free group is infinite. This result was generalized by Bardakov and Tolstykh [BT06] who proved that all free products, except $\mathbb{Z}_2 * \mathbb{Z}_2$, have infinite palindromic width. Following these investigations in free groups, Gilman and Keen [GK09] applied geometry of palindromes in two-generator free group to obtain discreteness conditions for two-generator subgroups in $SL(2, \mathbb{C})$. Palindromes in groups also appeared in the context of geometry of automorphisms of free groups, for example, see [GJ00].

Date: February 22, 2017.

2010 Mathematics Subject Classification. (Primary); 20F65 (Secondary) 20E06.

Key words and phrases. palindromic width, graph of groups, HNN extension, free product .

The intersection of the palindromic automorphism group and the IA-subgroup of a free group has analogy with the hyperelliptic Torelli subgroup of the mapping class groups. Motivated from this viewpoint, Fullarton [Ful15] investigated the Palindromic Torelli subgroup and obtained a generating set that normally generates the group. Fullarton then applied his result to obtain a finite presentation of the level 2 congruence subgroup. In [BGS15a], some algebraic properties of the palindromic automorphism group of a free group was investigated and using an algebraic approach, the authors proved that the palindromic Torelli subgroup is the intersection of the IA-automorphisms with the commutator of the elementary palindromic automorphism group.

Recently, there have been a series of work that aims to understand the palindromic widths in several other classes of groups including relatively free groups. Bardakov and Gongopadhyay have proved finiteness of palindromic widths of finitely generated free nilpotent groups and certain solvable groups, see [BG14b, BG14a, BG15]. In [BBG16], finiteness of palindromic width of nilpotent products has been proved. Palindromic widths of wreath products and Grigorchuk groups have been investigated by Fink [Fin17, Fin14]. Riley and Sale have investigated palindromic widths in certain wreath products and solvable groups [RS14] using finitely supported functions from \mathbb{Z}^r to the given group. Fink and Thom [FT15] have studied palindromic widths in simple groups and yielded the first examples of groups having finite palindromic widths but infinite commutator widths.

In this paper, we investigate palindromic width of the fundamental group of a graph of groups. We recall that a *graph of groups* (G, Y) consists of a non-empty, connected graph Y , a group G_P for each $P \in \text{vert } Y$ and a group G_e for each $e \in \text{edge } Y$, together with monomorphisms $G_e \rightarrow G_{\omega(e)}$, where for each edge e , $\alpha(e)$ is the initial vertex and $\omega(e)$ is the final vertex. We assume that $G_e = G_{\bar{e}}$. For each $P \in \text{vert } Y$, let S_P be the generating set of G_P . Also, let T be a maximal tree in Y . We fix

$$S = \cup_{P \in \text{vert } Y} S_P \cup \{e \in \text{edge}(Y - T)\}$$

to be the *standard generating set* of the fundamental group of (G, Y) , $\pi_1(G, Y)$. We prove the following theorem.

Theorem 1.1. *Let Y be a non-empty, connected graph. Let $\pi_1(G, Y)$ be the fundamental group of the graph of groups of Y with the standard generating set S . Then the palindromic width of $\pi_1(G, Y)$ is infinite if*

- (1) Y is a loop with a vertex P and edge e ; and the image of G_e is a proper subgroup of G_P ; or
- (2) Y has more than one vertex and has a circuit; or
- (3) Y is a tree and has an oriented edge $e = [P_1, P_2]$ such that removing e , while retaining P_1 and P_2 , gives two disjoint graphs Y_1 and Y_2 with $P_i \in \text{vert } Y_i$ satisfying the following: extending $G_e \rightarrow G_{P_i}$ to $\phi_i : G_e \rightarrow \pi_1(G, Y_i)$, $i = 1, 2$, we get $[\pi_1(G, Y_1) : \phi_1(G_e)] \geq 3$ and $[\pi_1(G, Y_2) : \phi_2(G_e)] \geq 2$.

In particular, we prove that the palindromic widths of HNN extensions of groups by proper associated subgroups is infinite and also, the amalgamated free products of two groups is infinite (except when the amalgamated subgroup has index two in each

of the factors), thus answering Question 3 and Question 4 in [BG15], see also Problem 6 and Problem 7 in [BGSVW15].

2. MAIN IDEA OF THE PROOF

Let G be the group under consideration. An element g in G is a *group-palindrome* if g is equal to its reverse word \bar{g} . This notion is weaker than the notion of ‘word-palindromes’, see [BBG16] for a comparison of these two notions. The set \mathcal{P} of word-palindromes is obviously a subset of \mathcal{GP} , the set of group-palindromic words. Thus, for an element g in G , $l_{\mathcal{GP}}(g) \leq l_{\mathcal{P}}(g)$. Consequently, the palindromic width with respect to group-palindromes does not exceed $pw(G, S)$. We shall show that the palindromic width with respect to group-palindromes is infinite and that will establish the main results. To achieve this, we shall use quasi-morphism techniques.

Definition 2.1. Let H be a group. A map $\Delta : H \rightarrow \mathbb{R}$ is called a *quasi-homomorphism* if there exists a constant c such that $\Delta(xy) \leq \Delta(x) + \Delta(y) + c$.

Quasi-morphisms have wide applications in mathematics and for a brief survey on these objects see [Kot04]. However, our motivation for using them in this paper comes from the work of Bardakov [Bar97] and Dobrynina [Dob00, Dob09] where the authors have proved infiniteness of verbal subgroups of HNN-extensions and amalgamated free products, also see [BST05], [BT06].

It will be shown in this paper that the quasi-homomorphism will be bounded if G has finite palindromic width. But, then we shall construct a sequence of elements in our groups where Δ will be bounded away and this will establish the results. Using quasi-homomorphisms, we prove that the palindromic widths of HNN extensions and amalgamated free products are infinite. As soon as these results are established, Theorem 1.1 will follow observing that the fundamental group of a graph of group is either an HNN extension or an amalgamated free product. We divide the proof of Theorem 1.1 into three sections. The case of HNN extensions has been studied in Section 3.1, the amalgamated free product case has been worked out in Section 4. In the final section, Theorem 1.1 has been derived.

Notation. Let f and g be functions over non-zero integers. We write $f =_m g$ to denote that $f(k) = g(k)$ for all values of k except at most m values.

Then clearly, $f =_m g$ and $g =_n h$ implies $f =_{m+n} h$. Also $f =_m g$ and $f' =_n g'$ implies $f + f' =_{m+n} g + g'$.

3. PALINDROMIC WIDTH OF HNN EXTENSION

Let G be a group and A and B be proper isomorphic subgroups of G with the isomorphism $\phi : A \rightarrow B$. Then the HNN extension of G is

$$G_{*A \cong B} = G* = \langle G, t \mid t^{-1}at = \phi(a), a \in A \rangle.$$

In this section we prove the following.

Proposition 3.1. Let G be a group with a generating set S . Let A and B be proper isomorphic subgroups of G and $\phi : A \rightarrow B$ be an isomorphism. The HNN extension

$$G* = \langle G, t \mid t^{-1}at = \phi(a), a \in A \rangle$$

of G with associated subgroups A and B has infinite palindromic width with respect to the generating set $S \cup \{t, t^{-1}\}$.

A sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \dots, g_{n-1}, t^{\epsilon_n}, g_n$, $n \geq 0$, is said to be *reduced* if it does not contain subsequences of the form t^{-1}, g_i, t with $g_i \in A$ or t, g_i, t^{-1} with $g_i \in B$. By Britton's Lemma, if a sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \dots, g_{n-1}, t^{\epsilon_n}, g_n$ is reduced and $n \geq 1$, then $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_n} g_n$ is not trivial in G^* and we call it a *reduced word*.

Such a representation of a group element of an HNN extension is not unique but the following lemma holds:

Lemma 3.2. *Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_n} g_n$ and $h = h_0 t^{\theta_1} h_1 t^{\theta_2} \dots h_{m-1} t^{\theta_m} h_m$ be reduced words and suppose $g = h$ in G^* . Then $m = n$ and $\epsilon_i = \theta_i$ for $i = 0, 1, \dots, n$.*

Proof. Since $g = h$, we have

$$1 = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n h_m^{-1} t^{-\theta_m} \dots t^{-\theta_1} h_0^{-1}.$$

Since g and h are reduced, we require $g_n h_m^{-1}$ to belong to A or B depending on the signs of ϵ_n and θ_m and also $\epsilon_n = \theta_m$. For further reduction, we need $\epsilon_i = \theta_i$ and $n = m$. \square

Definition 3.3. The *signature* of $g \in G^*$ is the sequence $sqn(g) = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, $\epsilon_i \in \{1, -1\}$ for $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_n} g_n$.

By Lemma 3.2, the signature of any $g \in G^*$ is unique, irrespective of the choice of the reduced word.

Let $\sigma = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ be a signature. Then the length of the signature $|\sigma| = n$. And the inverse signature, $\sigma^{-1} = (-\epsilon_n, -\epsilon_{n-1}, \dots, -\epsilon_1)$. So, $sqn(g^{-1}) = (sqn(g))^{-1}$.

Product of two signatures σ and τ , $\sigma\tau$, is obtained by writing τ after σ .

Suppose $\sigma = \sigma_1 \rho$ and $\tau = \rho^{-1} \tau_1$ with $|\rho| = r$, then we can define an r -product,

$$\sigma[r]\tau = \sigma_1 \tau_1.$$

The following lemma is immediate from the above notions.

Lemma 3.4. *For any $g, h \in G^*$, there exists an integer $r \geq 0$ such that $sqn(gh) = sqn(g)[r]sqn(h)$, with $sqn(g) = \sigma_1 \rho$ and $sqn(h) = \rho^{-1} \tau_1$ and $|\rho| = r$.*

A reduced expression is called *positive (negative)* if all exponents ϵ_i are positive (resp. negative). Further, if it is either positive or negative then the reduced expression is called *homogeneous*.

3.1. Proof of Proposition 3.1.

Let $\sigma = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ be the signature of an $g \in G^*$. We define,

$p_k(g)$ = number of $+1, +1, \dots, +1$ sections of length k ;

$m_k(g)$ = number of $-1, -1, \dots, -1$ sections of length k ;

$d_k(g) = p_k(g) - m_k(g)$;

$r_k(g)$ = remainder of $d_k(g)$ divided by 2; and,

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g).$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in G^*$.

Lemma 3.5. *For any elements $g, h \in G^*$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 6$, i.e. Δ is a quasi-homomorphism.*

Proof. Put $\sigma = sqn(g)$ and $\tau = sqn(h)$. By Lemma 1, we have, $\sigma = \sigma_1\rho$, $\tau = \rho^{-1}\tau_1$, $|\rho| = r \geq 0$ and $sqn(gh) = \sigma[r]\tau = \sigma_1\tau_1$.

For any two signatures α and β , $p_k(\alpha\beta) =_1 p_k(\alpha) + p_k(\beta)$.

Now,

$$p_k(gh) = p_k(\sigma_1\tau_1) =_1 p_k(\sigma_1) + p_k(\tau_1)$$

Further,

$$p_k(g) = p_k(\sigma_1\rho) =_1 p_k(\sigma_1) + p_k(\rho)$$

$$p_k(h) = p_k(\rho^{-1}\tau_1) =_1 p_k(\rho^{-1}) + p_k(\tau_1)$$

Hence, we have,

$$p_k(gh) =_3 p_k(g) - p_k(\rho) + p_k(h) - p_k(\rho^{-1}).$$

As $p_k(\rho^{-1}) = m_k(\rho)$, using the definition of d_k , we have

$$d_k(gh) =_6 d_k(g) + d_k(h).$$

Then we have $r_k(gh) =_6 r_k(g) + r_k(h)$, and so,

$$\Delta(gh) \leq \Delta(g) + \Delta(h) + 6.$$

Therefore, Δ is a quasi-homomorphism. \square

Definition 3.6. Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_{n-1}} g_n$ be a reduced element in G^* . Put

$$\bar{g} = \bar{g}_n t^{\epsilon_{n-1}} \bar{g}_{n-1} t^{\epsilon_{n-2}} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0.$$

We say g is a group-palindrome if $\bar{g} = g$.

Lemma 3.7. *A group-palindrome $g \in G^*$ has the form*

$$g = \begin{cases} g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} g'_k t^{\epsilon_k} \bar{g}_{k-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0, & \text{if } |sqn(g)| = 2k \\ g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} g'_k t^{\epsilon_{k+1}} \bar{g}_k t^{\epsilon_k} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0, & \text{if } |sqn(g)| = 2k + 1 \end{cases}$$

where $g'_k = g_k x$ where $x \in A \cup B$.

Proof. CASE 1: $|sqn(g)|$ is odd.

For $|sqn(g)| = 1$:

Let $g = g_0 t^{\epsilon_1} g_1$

Since g is a palindrome, $\bar{g} = g$.

So,

$$\begin{aligned} g\bar{g}^{-1} &= 1 \\ g_0 t^{\epsilon_1} g_1 \bar{g}_0^{-1} t^{-\epsilon_1} \bar{g}_1^{-1} &= 1 \end{aligned}$$

Since this expression is not reduced, $g_1\bar{g}_0^{-1} = a \in A$, with $t^{\epsilon_1}at^{-\epsilon_1} = b \in B$, say.

$$\begin{aligned} g_0b\bar{g}_1^{-1} &= 1 \\ \bar{g}_1 &= g_0b \\ (3.1) \quad g_1 &= \bar{b}\bar{g}_0 = a\bar{g}_0 \end{aligned}$$

So,

$$g = g_0t^{\epsilon_1}a\bar{g}_0 = g_0bt^{\epsilon_1}\bar{g}_0$$

Now, $\bar{g} = g_0t^{\epsilon_1}\bar{b}\bar{g}_0$

Using (3.1), $\bar{g} = g_0t^{\epsilon_1}\bar{b}\bar{g}_0 = g_0t^{\epsilon_1}a\bar{g}_0$.

For $|sqn(g)| = 3$:

Let $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2}g_2t^{\epsilon_3}g_3$

Since g is a palindrome, $\bar{g} = g$.

So,

$$g\bar{g}^{-1} = 1$$

Also, we get, $\epsilon_1 = \epsilon_3$.

$$g_0t^{\epsilon_1}g_1t^{\epsilon_2}g_2t^{\epsilon_1}g_3\bar{g}_0^{-1}t^{-\epsilon_1}\bar{g}_1^{-1}t^{-\epsilon_2}\bar{g}_2^{-1}t^{-\epsilon_1}\bar{g}_3^{-1} = 1$$

Since this expression is not reduced, $g_3\bar{g}_0^{-1} = a_1 \in A$, with $t^{\epsilon_1}a_1t^{-\epsilon_1} = b_1 \in B$, say.

$$g_0t^{\epsilon_1}g_1t^{\epsilon_2}g_2b_1\bar{g}_1^{-1}t^{-\epsilon_2}\bar{g}_2^{-1}t^{-\epsilon_1}\bar{g}_3^{-1} = 1$$

Further, let $g_2b_1\bar{g}_1^{-1} = b_2 \in B$, with $t^{\epsilon_2}b_2t^{-\epsilon_2} = a_2 \in A$, say.

$$g_0t^{\epsilon_1}g_1a_2\bar{g}_2^{-1}t^{-\epsilon_1}\bar{g}_3^{-1} = 1$$

Let $g_1a_2\bar{g}_2^{-1} = a_3 \in A$, with $t^{\epsilon_1}a_3t^{-\epsilon_1} = b_3 \in B$.

$$g_0b_3\bar{g}_3^{-1} = 1$$

We get,

$$\bar{g}_3 = g_0b_3$$

Using this and the relations we obtained above, we can compute

$$\begin{aligned} g &= g_0t^{\epsilon_1}g_1a_2t^{\epsilon_2}\bar{g}_1t^{\epsilon_1}\bar{g}_0 = g_0t^{\epsilon_1}g_1b_2t^{\epsilon_2}\bar{g}_1t^{\epsilon_1}\bar{g}_0 \\ \bar{g} &= g_0t^{\epsilon_1}g_1t^{\epsilon_2}b_2\bar{g}_1t^{\epsilon_1}\bar{g}_0. \end{aligned}$$

So, $\bar{g} = g_0t^{\epsilon_1}g_1a_2t^{\epsilon_2}\bar{g}_1t^{\epsilon_1}\bar{g}_0 = g$.

Now suppose for $|sqn(g)| = 2k + 1$.

And

$$g = g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_k}g_kxt^{\epsilon_{k+1}}\bar{g}_kt^{\epsilon_k} \dots \bar{g}_1t^{\epsilon_1}\bar{g}_0$$

where $x \in A$ or B .

For $|sqn(g)| = 2k + 3$,

$$g = g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_k}g_kt^{\epsilon_{k+1}}g_{k+1} \dots t^{\epsilon_{2k+3}}g_{2k+3}$$

$$g = g_0t^{\epsilon_1}ht^{\epsilon_{2k+3}}g_{2k+3}$$

where h is a palindrome of length $2k + 1$.

Now since $\bar{g} = g$, we will get $g_{2k+3} = x\bar{g}_0$, where $x \in A$ or B .

Then, by induction, we can see that

$$g = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_{k+1}} g_{k+1} x' t^{\epsilon_{k+2}} \bar{g}_{k+1} t^{\epsilon_{k+1}} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$$

where $x' \in A$ or B .

CASE 2: $|sqn(g)|$ is even.

For $|sqn(g)| = 2$:

Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2$

Since g is a palindrome, $\bar{g} = g$.

So,

$$g\bar{g}^{-1} = 1$$

Also, we get, $\epsilon_1 = \epsilon_2$.

$$g_0 t^{\epsilon_1} g_1 t^{\epsilon_1} g_2 \bar{g}_0^{-1} t^{-\epsilon_1} \bar{g}_1^{-1} t^{-\epsilon_1} \bar{g}_2^{-1} = 1$$

Since this expression is not reduced, $g_2 \bar{g}_0^{-1} = a_1 \in A$, with $t^{\epsilon_1} a_1 t^{-\epsilon_1} = b_1 \in B$, say.

$$g_0 t^{\epsilon_1} g_1 b_1 \bar{g}_1^{-1} t^{-\epsilon_1} \bar{g}_2^{-1} = 1$$

Further, let $g_1 b_1 \bar{g}_1^{-1} = a_2 \in A$, with $t^{\epsilon_1} a_2 t^{-\epsilon_1} = b_2 \in B$, say.

$$g_0 b_2 \bar{g}_2^{-1} = 1$$

We get,

$$\bar{g}_2 = g_0 b_2$$

Using this and the relations we obtained above, we can compute

$$g = g_0 t^{\epsilon_1} g_1 b_1 t^{\epsilon_1} \bar{g}_0 = g_0 t^{\epsilon_1} g_1 \bar{a}_2 t^{\epsilon_1} \bar{g}_0$$

Also, we can easily see that $\bar{g} = g$.

For $|sqn(g)| = 4$:

Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 t^{\epsilon_3} g_3 t^{\epsilon_4} g_4$

Since g is a palindrome, $\bar{g} = g$.

So,

$$g\bar{g}^{-1} = 1$$

Also, we get, $\epsilon_1 = \epsilon_4$ and $\epsilon_2 = \epsilon_3$.

$$g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 t^{\epsilon_2} g_3 t^{\epsilon_1} g_4 \bar{g}_0^{-1} t^{-\epsilon_1} \bar{g}_1^{-1} t^{-\epsilon_2} \bar{g}_2^{-1} t^{-\epsilon_2} \bar{g}_3^{-1} t^{-\epsilon_1} \bar{g}_4^{-1} = 1$$

Since this expression is not reduced, $g_4 \bar{g}_0^{-1} = a_1 \in A$, with $t^{\epsilon_1} a_1 t^{-\epsilon_1} = b_1 \in B$, say.

$$g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 t^{\epsilon_2} g_3 b_1 \bar{g}_1^{-1} t^{-\epsilon_2} \bar{g}_2^{-1} t^{-\epsilon_1} \bar{g}_3^{-1} t^{-\epsilon_1} \bar{g}_4^{-1} = 1$$

Further, let $g_3 b_1 \bar{g}_1^{-1} = b_2 \in B$, with $t^{\epsilon_2} b_2 t^{-\epsilon_2} = a_2 \in A$, say.

$$g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 a_2 \bar{g}_2^{-1} t^{-\epsilon_1} \bar{g}_3^{-1} t^{-\epsilon_1} \bar{g}_4^{-1} = 1$$

Proceeding in this manner we get,

$$g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 a_2 t^{\epsilon_2} \bar{g}_1 t^{\epsilon_1} \bar{g}_0 = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \bar{a}_3 t^{\epsilon_2} \bar{g}_1 t^{\epsilon_1} \bar{g}_0$$

Also, $\bar{g} = g$.

Now suppose for $|sqn(g)| = 2k$.

And

$$g = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} g_k x t^{\epsilon_k} \bar{g}_{k-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$$

where $x \in A$ or B .

For $|sqn(g)| = 2k + 2$,

$$g = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_k} g_k t^{\epsilon_{k+1}} g_{k+1} \dots t^{\epsilon_{2k+2}} g_{2k+2}$$

$$g = g_0 t^{\epsilon_1} h t^{\epsilon_{2k+2}} g_{2k+2}$$

where h is a palindrome of length $2k$.

Now since $\bar{g} = g$, we will get $g_{2k+2} = x \bar{g}_0$, where $x \in A$ or B .

Then, by induction, we can see that

$$g = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_{k+1}} g_{k+1} x' t^{\epsilon_{k+1}} \bar{g}_k \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$$

where $x' \in A$ or B . □

Lemma 3.8. *Let $p \in G^*$ be a group-palindrome. Then, $\Delta(p) \leq 1$.*

Proof. Let p be a group-palindrome in G^* of non-zero length.

Then p can be represented as $p = uv\bar{u}$, where v is the maximal homogeneous palindromic sub-word in p and \bar{u} is u written in reverse.

For example, if $p = g_0 t^{\epsilon_1} g_1 \dots t^{-1} g_i t g_{i+1} t \dots t \bar{g}_{i+1} t \bar{g}_i t^{-1} \bar{g}_{i-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$,

$$u = g_0 t^{\epsilon_1} g_1 \dots t^{-1};$$

$$v = g_i t g_{i+1} t \dots t \bar{g}_{i+1} t \bar{g}_i;$$

$$\bar{u} = t^{-1} \bar{g}_{i-1} \dots \bar{g}_1 t^{\epsilon_1} \bar{g}_0.$$

Then for every k , $d_k(u) = d_k(\bar{u})$. As v is homogeneous, if k' is the length of $sqn(v)$, then

$$p_{k'}(p) = 2p_{k'}(u) + p_{k'}(v), \text{ or } m_{k'}(p) = 2m_{k'}(u) + m_{k'}(v).$$

For all other k , $p_k(p) = 2p_k(u)$ and $m_k(p) = 2m_k(u)$.

Therefore,

$$r_{k'}(p) = 1, \text{ and } r_k(p) = 0 \text{ for all other } k.$$

Thus,

$$\Delta(p) = 1.$$

If $p \in G$, then $\Delta(p) = 0$. So, $\Delta(p) \leq 1$. □

Then, if $g \in G^*$ is a product of k group-palindromes, say $g = p_1 p_1 \dots p_k$, then

$$(3.2) \quad \Delta(g) = \Delta(p_1 p_1 \dots p_k) \leq \Delta(p_1) + \Delta(p_2) + \dots + \Delta(p_k) + 6(k-1) \leq 7k - 6.$$

Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence $\{a_i\}$ for which $\Delta(a_i)$ is increasing.

$$\text{Let } a_1 = g_0 t g_1 t^{-1} g_2 t g_3$$

$$d_1(a_1) = 1, \text{ so } \Delta(a_1) = 1.$$

For $a_2 = g_0tg_1t^{-1}g_2tg_3t^{-1}g_4t^{-1}g_5tg_6tg_7t^{-1}g_8t^{-1}g_9$
 $d_1(a_2) = 1, d_2(a_2) = -1$ so $\Delta(a_2) = 2$.

Let $a_3 = g_0tg_1t^{-1}g_2tg_3t^{-1}g_4t^{-1}g_5tg_6tg_7t^{-1}g_8t^{-1}g_9tg_{10}tg_{11}tg_{12}t^{-1}g_{13}t^{-1}g_{14}t^{-1}g_{15}tg_{16}tg_{17}tg_{18}$
 $d_1(a_3) = 1, d_2(a_3) = -1, d_3(a_3) = 1$, so, $\Delta(a_3) = 3$.

Constructing elements $a_4, a_5, \dots, a_n, \dots$ in a similar fashion, we get

$$\Delta(a_n) = n.$$

Then, by (3.2), we get that the palindromic width of G^* is infinite. This proves Proposition 3.1.

4. PALINDROMIC WIDTH OF AMALGAMATED FREE PRODUCT

In this section we shall prove the following.

Proposition 4.1. Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C and $|A : C| \geq 3, |B : C| \geq 2$. Then $pw(G, \{A, B\})$ is infinite.

To prove this, we shall divide the theorem into three cases. First recall the notion of free product with amalgamation.

Let $A = \langle a_1, \dots | R_1, \dots \rangle$ and $B = \langle b_1, \dots | S_1, \dots \rangle$ be groups. Let $C_1 \subset A$ and $C_2 \subset B$ be subgroups such that there exists an isomorphism $\phi : C_1 \rightarrow C_2$. Then the free product of A and B , amalgamating the subgroups C_1 and C_2 by the isomorphism ϕ is the group

$$G = \langle a_1, \dots, b_1, \dots | R_1, \dots, S_1, \dots, c = \phi(c), c \in C_1 \rangle$$

We can view G as the quotient of the free product $A * B$ by the normal subgroup generated by $\{c\phi(c)^{-1} | c \in C_1\}$. The subgroups A and B are called factors of G , and since C_1 and C_2 are identified in G , we will denote them both by C .

4.1. CASE 1. For any $a \in A$ and $b \in B$, $aC = a^{-1}C$ and $bC = b^{-1}C$.

Proposition 4.2. Let G be a group and H be a subgroup of G . If for any $g \in G$, $g^{-1}H = gH$ then, H is a normal subgroup of G .

Proof. Given, for every $g \in G, g^{-1}H = gH$. That is, $g^2 \in H$.

For every $g \in G$,

$$g^{-1}Hg = gHg.$$

So, for $g \in G, h \in H$, there exists $h_1 \in H$ such that

$$g^{-1}hg = gh_1g = (gh_1)^2h_1^{-1} \in H.$$

Thus, we have $g^{-1}Hg = H$. □

So, C is a normal subgroup of A and B . Let T_1 be a system of representatives of right cosets of C in A and T_2 be that of C in B .

Definition 4.3. A C -normal form is a sequence x_0, x_1, \dots, x_n such that

- (1) $x_0 \in C$.
- (2) $x_i \in T_1 - \{1\}$ or $x_i \in T_2 - \{1\}$ for $i \geq 1$.
- (3) Consecutive terms x_i and x_{i+1} lie in distinct systems of representatives.

Now, for any element $g \in G$ there corresponds a C -normal form x_0, x_1, \dots, x_n and g can be uniquely written as $x_0 x_1 \dots x_n$.

Lemma 4.4. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C . Let $|A : C| \geq 3$, $|B : C| \geq 2$ and C is normal in both A and B . Then $pw(G, \{A, B\})$ is infinite..*

Proof. If C is a normal subgroup of A and B , we can take $T_1 = A/C$ and $T_2 = B/C$. We have $|A/C| \geq 3$ and $|B/C| \geq 2$. Consider the map

$$f : G \rightarrow (A/C) * (B/C)$$

given by $f(x_0 x_1 \dots x_n) = x_1 \dots x_n$. It is easy to see that this is a surjective homomorphism. Hence, by [BG14b, Lemma 2.3],

$$pw(G) \geq pw((A/C) * (B/C)).$$

By the result of Bardakov and Tolstykh [BT06], the palindromic width of $(A/C) * (B/C)$ is infinite. Thus, we have the required result. \square

4.2. CASE 2. There exists at least a non-trivial element $a \in A \cup B$ for which $aC \neq a^{-1}C$. We can further divide this into two cases:

4.2.1. CASE 2.1. For a non-trivial $a \in A \cup B$ such that $aC \neq a^{-1}C$, we have $CaC \neq Ca^{-1}C$. Then we shall prove the following:

Lemma 4.5. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C . Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A$ for which $CaC \neq Ca^{-1}C$. Then $pw(G, \{A, B\})$ is infinite..*

To prove this, we shall use the quasi-homomorphism constructed in [Dob00, Dob09].

Definition 4.6. A sequence x_1, \dots, x_n , $n \geq 0$, is said to be *reduced* if

- (1) Each x_i is in one of the factors.
- (2) Successive x_i, x_{i+1} come from different factors.
- (3) If $n > 1$, no x_i is in C .
- (4) If $n = 1$, $x_1 \neq 1$.

By the Normal Form Theorem for free products with amalgamation, see for eg. [LS01], if x_1, \dots, x_n is a reduced sequence, $n \geq 1$, then the product $x_1 \dots x_n \neq 1$ is in G and it is called a reduced word. Such a representation of a group element is not unique but the following lemma holds:

Proposition 4.7. Let $g = x_1 \dots x_n$ and $h = y_1 \dots y_m$ be reduced words such that $g = h$ in G . Then $m = n$.

Proof. Since $g = h$, we have

$$1 = x_1 \dots x_n y_m^{-1} \dots y_2^{-1} y_1^{-1}.$$

Since g and h are reduced, we require $x_n y_m^{-1}$ to belong to C . To reduce it further we need $x_{n-1} x_n y_m^{-1} y_{m-1}^{-1}$ to be in C and so on. Hence, $m = n$. \square

So, if for $g \in G$, $g = x_1 \dots x_n$ is a reduced expression, we define the *length* of g to be, $l(g) = n$.

Definition 4.8. Assume that $a \in A$.

Let $g \in G$ and $g = x_1 x_2 \dots x_n$ be a reduced word representing g . Then if $x_i \in A$ and $x_i = ua^\epsilon u'$ where $\epsilon \in \{+1, -1\}$, $u, u' \in C$, we replace x_i by $ua^\epsilon u'$ and join u to x_{i-1} to get x'_{i-1} and, u' to x_{i+1} to get x'_{i+1} . Then we get

$$g = x_1 x_2 \dots x'_{i-1} a^\epsilon x'_{i+1} \dots x_n.$$

If $i = 1$, we join u' to x_2 as above and preserve u , and get

$$g = ua^\epsilon x'_2 \dots x_n.$$

If $i = n$, we join u to x_{n-1} and preserve u' to get

$$g = x_1 x_2 \dots x'_{n-1} a^\epsilon u'.$$

Such a representation is called the *special form* of g .

An *a-segment* of length $2k - 1$ is a segment of the reduced word of the following form

$$ax_1 \dots x_{2k-1}a$$

where $x_j \neq a$ for $j = 1, \dots, 2k - 1$ such that the length of $x_1 \dots x_{2k-1}$ is $2k - 1$.

For $g \in G$ expressed in special form, we define

$p_k(g)$ = number of a -segments of length $2k - 1$;

$m_k(g)$ = number of a^{-1} -segments of length $2k - 1$;

$d_k(g) = p_k(g) - m_k(g)$;

$r_k(g)$ = remainder of $d_k(g)$ divided by 2; and

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g).$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in G$.

Lemma 4.9. For any elements $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$, i.e. Δ is a quasi-homomorphism.

Proof. The proof follows from [Dob00, Lemma 2]. \square

Definition 4.10. Let $g = x_1 \dots x_n$ be a reduced word of $g \in G$. The elements x_k are said to be *syllables* of g . Let \bar{g} be the word obtained by writing g in the reverse order, i.e. $\bar{g} = x_n \dots x_1$. This is a non-trivial element of G . We say g is a group-palindrome if $\bar{g} = g$.

Remark 4.11. We are computing the palindromic width of G with respect to $A \cup B$. So we fix $A \cup B$ to be the generating set of G . And therefore, every syllable is a palindrome.

Lemma 4.12. *A group-palindrome $g \in G$ has the form*

$$g = \begin{cases} x_1 x_2 \dots x'_k x_k x_{k-1} \dots x_1, & \text{if } l(g) = 2k \\ x_1 x_2 \dots x_k x'_{k+1} x_k x_{k-1} \dots x_1, & \text{if } l(g) = 2k + 1 \end{cases}$$

where $x'_i = x_i c$ with $c \in C$.

Proof. CASE 1: $l(g)$ is odd.

The case of $l(g) = 1$ is trivial.

For $l(g) = 3$.

Let $g = x_1 x_2 x_3$. Since g is a palindrome, $\bar{g} = g$.

So,

$$\begin{aligned} g\bar{g}^{-1} &= 1 \\ x_1 x_2 x_3 x_1^{-1} x_2^{-1} x_3^{-1} &= 1 \end{aligned}$$

Since this expression is not reduced,

$$\begin{aligned} (4.1) \quad x_3 x_1^{-1} &= c_1 \in C, \text{ say.} \\ x_1 x_2 c_1 x_2^{-1} x_3^{-1} &= 1 \end{aligned}$$

Let

$$(4.2) \quad x_2 c_1 x_2^{-1} = c_2 \in C.$$

So,

$$\begin{aligned} x_1 c_2 x_3^{-1} &= 1 \\ x_3 &= x_1 c_2. \end{aligned}$$

Then we get,

$$g = x_1 (x_2 c_1) x_1.$$

So we have $\bar{g} = x_1 c_1 x_2 x_1$.

Using (4.1), (4.2), $\bar{g} = x_1 x_2 c_2 x_1 = x_1 x_2 c_1 x_1$.

For $l(g) = 5$:

Let $g = x_1 x_2 x_3 x_4 x_5$.

Since g is a palindrome, $\bar{g} = g$.

So,

$$\begin{aligned} g\bar{g}^{-1} &= 1; \\ x_1 x_2 x_3 x_4 x_5 x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} &= 1. \end{aligned}$$

Since this expression is not reduced, $x_5 x_1^{-1} = c_1 \in C$, say.

$$x_1 x_2 x_3 x_4 c_1 x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} = 1.$$

Further, let $x_4 c_1 x_2^{-1} = c_2 \in C$, say.

$$x_1 x_2 x_3 c_2 x_3^{-1} x_4^{-1} x_5^{-1} = 1.$$

Let $x_3c_2x_3^{-1} = c_3 \in C$.

$$x_1x_2c_3x_4^{-1}\bar{x}_5^{-1} = 1$$

Let $x_2c_3x_4^{-1} = c_4 \in C$.

$$x_1c_4x_5^{-1} = 1.$$

We get $x_5 = x_1c_4$.

$$g = x_1x_2(x_3c_2)x_2x_1.$$

Also, using the relations we obtained above, we can see that $c_1 = c_4$ and $c_2 = c_3$.
So,

$$\bar{g} = x_1x_2c_2x_3x_2x_1 = x_1x_2x_3c_3x_2x_1 = x_1x_2x_3c_2x_2x_1.$$

Now for $l(g) = 2k + 1$, assume

$$g = x_1x_2 \dots x_k(x_{k+1}c)x_k \dots x_2x_1$$

where $c \in C$.

For $l(g) = 2k + 3$,

$$g = x_1x_2 \dots x_{2k+3}$$

$$g = x_1hx_{2k+3}$$

where h is a palindrome of length $2k + 1$.

Now since $\bar{g} = g$, we will get $x_{2k+3} = c_1x_1$, where $c_1 \in C$.

Then, by induction, we can see that

$$g = x_1x_2 \dots x_{k+1}(x_{k+2}c')x_{k+1}x_k \dots x_1$$

where $c' \in C$.

CASE 2: $l(g)$ is even.

For $l(g) = 2$:

Let $g = x_1x_2$

Since g is a palindrome, $\bar{g} = g$.

So,

$$gg^{-1} = 1;$$

$$x_1x_2x_1^{-1}x_2^{-1} = 1.$$

Since this expression is not reduced, $x_2x_1^{-1} = c_1 \in C$, say.

$$x_1c_1x_2^{-1} = 1.$$

We get,

$$x_2 = x_1c_1.$$

Using this and the relations we obtained above, we can compute

$$g = (x_1c_1)x_1.$$

Also, we can easily see that $\bar{g} = x_1c_1x_1 = g$.

For $l(g) = 4$:

Let $g = x_1x_2x_3x_4$.

Since g is a palindrome, $\bar{g} = g$.

So,

$$\begin{aligned} g\bar{g}^{-1} &= 1; \\ x_1x_2x_3x_4x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} &= 1. \end{aligned}$$

Since this expression is not reduced, $x_4x_1^{-1} = c_1 \in C$, say.

$$x_1x_2x_3c_1x_2^{-1}x_3^{-1}x_4^{-1} = 1.$$

Further, let $x_3c_1x_2^{-1} = c_2 \in C$, say.

$$x_1x_2c_2x_3^{-1}x_4^{-1} = 1.$$

Let $x_2c_2x_3^{-1} = c_3 \in C$.

$$x_1c_3x_4^{-1} = 1.$$

We get $x_4 = x_1c_3$. Using this and the relations we obtained above, we can compute

$$g = x_1(x_2c_2)x_2x_1.$$

Clearly, $\bar{g} = x_1x_2c_2x_2x_1 = g$.

Now suppose for $l(g) = 2k$.

Assume

$$g = x_1x_2 \dots (x_kc)x_kx_{k-1} \dots x_1,$$

where $c \in C$.

For $l(g) = 2k + 2$,

$$g = x_1x_2 \dots x_{2k+1}x_{2k+2}.$$

$$g = x_1hx_{2k+2},$$

where h is a palindrome of length $2k$.

Now since $\bar{g} = g$, we will get $x_{2k+2} = c_1x_1$, where $c_1 \in C$.

Then, by induction, we can see that

$$g = x_1x_2 \dots x_k(x_{k+1}c')x_kx_{k-1} \dots x_1,$$

where $c' \in C$.

□

Lemma 4.13. *Let $p \in G$ be a group-palindrome. Then, $\Delta(p) \leq 3$.*

Proof. Let p be a group-palindrome in G of non-zero length.

Then p can be expressed as $p = hu\bar{h}$, where \bar{h} is h written in reverse.

Then, for every k , $d_k(h) = d_k(\bar{h})$.

If $u = a$, we have

$$\begin{aligned} p_k(p) &= {}_2 2p_k(h) \\ m_k(p) &= {}_1 2m_k(h) \end{aligned}$$

Then we get

$$d_k(p) = {}_3 2d_k(h).$$

If $u = a^{-1}$, as above, we get $d_k(p) =_3 2d_k(h)$.

If $u \neq a, a^{-1}$, we get $d_k(p) =_2 2d_k(h)$.

So, in general we have,

$$d_k(p) =_3 2d_k(h).$$

Thus,

$$r_k(p) =_3 0.$$

and hence, $\Delta(p) \leq 3$. □

Thus, if $g \in G$ is a product of k group-palindromes, say $g = p_1 p_1 \dots p_k$, then

$$(4.3) \quad \Delta(g) = \Delta(p_1 p_1 \dots p_k) \leq \Delta(p_1) + \Delta(p_2) + \dots + \Delta(p_k) + 9(k-1) \leq 12k - 9.$$

Proof of Lemma 4.5. Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence $\{g_i\}$ for which $\Delta(g_i)$ is increasing.

Let $b \in B$ but not in C .

$$\text{Let } g_1 = baba^{-1}ba;$$

$$p_1(g_1) = 0, p_2(g_1) = 1 \text{ and } p_k(g_1) = 0 \text{ for all other } k.$$

$$m_k(g_1) = 0 \text{ for all } k.$$

$$d_2(g_1) = 1 \text{ and } d_k(g_1) = 0 \text{ for all other } k.$$

$$\text{So, } \Delta(g_1) = 1.$$

$$\text{Let } g_2 = baba^{-1}baba^{-1}ba^{-1}ba; \text{ then,}$$

$$p_1(g_2) = 0, p_2(g_2) = p_3(g_2) = 1 \text{ and } p_k(g_2) = 0 \text{ for all other } k.$$

$$m_1(g_2) = m_2(g_2) = 1 \text{ and } m_k(g_2) = 0 \text{ for all other } k.$$

$$\text{So, } \Delta(g_2) = 2.$$

$$\text{Let } g_3 = baba^{-1}baba^{-1}ba^{-1}baba^{-1}ba^{-1}ba;$$

$$p_1(g_3) = 0, p_2(g_3) = p_3(g_3) = p_4(g_3) = 1 \text{ and } p_k(g_3) = 0 \text{ for all other } k.$$

$$m_1(g_3) = 3, m_2(g_3) = 2 \text{ and } m_k(g_3) = 0 \text{ for all other } k.$$

$$\text{So, } \Delta(g_3) = 4.$$

In general, for

$$g_n = baba^{-1} \dots ba(ba^{-1})^n ba$$

we have

$$p_1(g_n) = 0, p_k(g_n) = 1 \text{ for } 1 < k < n+1.$$

$$m_1(g_n) = \frac{n(n-1)}{2}, m_2(g_n) = n-1 \text{ and for } k \neq 1, 2, m_k(g_n) = 0.$$

We have

$$\Delta(g_n) = r_1 + r_2 + (n-1).$$

where r_1 is the remainder of $\frac{n(n-1)}{2}$ divided by 2 and r_2 is that of n divided by 2. So,

$$\Delta(g_n) \geq n-1.$$

Then, by (4.3), we get that the palindromic width of G is infinite. This proves Lemma 4.5.

4.2.2. CASE 2.2. For a non-trivial $a \in A \cup B$ such that $aC \neq a^{-1}C$, we have $CaC = Ca^{-1}C$.

Lemma 4.14. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C . Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A$ for which $CaC = Ca^{-1}C$. Then $pw(G, \{A, B\})$ is infinite.*

We use a method similar to Case 2.1 to prove this lemma. We define the special form for elements of G which is a simple modification of the special form used in Case 2.1.

Definition 4.15. Assume $a \in A$.

Let $g \in G$ and $g = x_1 x_2 \dots x_n$ is a reduced word representing g . If $x_i \in A$ and $x_i = u_3 a u_4, x_i = u_5 a^{-1} u_6$ where $u_3, u_4, u_5, u_6 \in C$, we fix one of the representations and we write x_i by $ua^\epsilon u'$ and replace x_i by $ua^\epsilon u'$ and join u to x_{i-1} to get x'_{i-1} and u' to x_{i+1} to get x'_{i+1} . Then we get

$$g = x_1 x_2 \dots x'_{i-1} a^\epsilon x'_{i+1} \dots x_n.$$

If $i = 1$, we join u' to x_2 as above and preserve u and get

$$g = ua^\epsilon x'_2 \dots x_n.$$

If $i = n$, we join u to x_{n-1} and preserve u' to get

$$g = x_1 x_2 \dots x'_{n-1} a^\epsilon u'.$$

We call this the *special form* of g .

We use the same notations as in Case 2.1.

Now, as in the previous case, we define the quasi-homomorphism.

Lemma 4.16. *For any elements $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$.*

For proof of the above lemma, see [Dob00, Dob09]. Further we have the following results which can be obtained similarly as in Case 2.1.

Lemma 4.17. *Let $p \in G$ be a group-palindrome. Then, $\Delta(p) \leq 3$.*

If $g \in G$ is a product of k group-palindromes, say $g = p_1 p_1 \dots p_k$, then

$$(4.4) \quad \Delta(g) = \Delta(p_1 p_1 \dots p_k) \leq \Delta(p_1) + \Delta(p_2) + \dots + \Delta(p_k) + 9(k-1) \leq 12k - 9.$$

And finally, for the same sequence in Case 2.1, using (4.4), we get that the palindromic width of G is infinite.

4.3. Proof of Proposition 4.1. The result follows by combining Lemma 4.4, Lemma 4.5 and Lemma 4.14.

So far we have shown that the palindromic width of $G = A *_C B$, when $|A : C| \geq 3$, $|B : C| \geq 2$, is infinite. Let's now consider the case when $|A : C| \leq 2$, $|B : C| \leq 2$.

Proposition 4.18. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C and $|A : C| \leq 3$, $|B : C| \leq 2$. Then $pw(G, \{A, B\})$ is finite.*

Proof. We only need to consider the case of $|A : C| = 2$, $|B : C| = 2$.

Then C is a normal subgroup of both A and B . As in Lemma 4.4, we have a surjective homomorphism from G onto the free product of the quotient groups $(A/C)*(B/C)$. Since $(A/C)*(B/C) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we have our proof. \square

5. PROOF OF THEOREM 1.1

5.1. Graph of Groups. First we recall the notion of graph of groups, some basic references are [Ser80], [DD89], [DK16].

Definition 5.1. A graph of groups (G, Y) consists of a non-empty, connected graph Y , a group G_P for each $P \in \text{vert } Y$ and a group G_e for each $e \in \text{edge } Y$, together with monomorphisms $G_e \rightarrow G_{\omega(e)}$, where for each edge e , $\alpha(e)$ is the initial vertex and $\omega(e)$ is the final vertex. We assume that $G_e = G_{\bar{e}}$.

Next, we define a group $F(G, Y)$ generated by the groups G_P and the elements $e \in \text{edge } Y$ with the following relations:

$$\bar{e} = e^{-1}$$

and

$$ea^e e^{-1} = a^{\bar{e}}$$

where $e \in \text{edge } Y$, $a \in G_e$ and a^e is the image of a under the above mentioned monomorphism.

Explicitly, $F(G, Y)$ is the free product of G_P and the free group generated over edge Y , quotiented by the normal closure of $\{e\bar{e}, ea^e e^{-1}(a^{\bar{e}})^{-1} | e \in \text{edge } Y, a \in G_e\}$.

Definition 5.2. Let c be a path in Y originating at a vertex P_0 with edges e_1, e_2, \dots, e_n and $P_i = \omega(e_i) = \alpha(e_{i+1})$. Then, a word of type c in $F(G, Y)$ is a pair (c, μ) where $\mu = (r_0, r_1, \dots, r_n)$ is a sequence with $r_i \in G_{P_i}$.

$$|c, \mu| = r_0 e_1 r_1 \dots e_n r_n \in F(G, Y)$$

is the element associated with (c, μ) .

5.1.1. Definitions of the fundamental group of (G, Y) .

1) Let P_0 be a vertex of Y . We define $\pi_1(G, Y, P_0)$, fundamental group of (G, Y) at P_0 , to be the set of all elements of $F(G, Y)$ of the type (c, μ) where c is a loop with P_0 as the initial and terminal vertex. This is a subgroup of $F(G, Y)$.

2) Let T be a maximal tree of Y . Then $\pi_1(G, Y, T)$, fundamental group of (G, Y) at T , is the quotient of $F(G, Y)$ by the normal closure of $\{e \in \text{edge } T\}$.

Let g_e denote the image of e in $\pi_1(G, Y, T)$. Thus, $\pi_1(G, Y, T)$ is generated by the groups G_P and the elements g_e subject to relations

$$g_e a^e g_e^{-1} = a^{\bar{e}}$$

$$g_{\bar{e}} = g_e^{-1} \text{ if } e \in \text{edge } Y, a \in G_e.$$

$$g_e = 1 \text{ if } e \in \text{edge } T.$$

The fundamental group G of a graph of groups satisfy the following properties (see [DK16]):

1. There is a collection of compatible homomorphisms $G_v \rightarrow G$, $G_e \rightarrow G$, $v \in \text{vert}(Y)$, $e \in \text{edge}(Y)$, so that whenever $v \in \{\alpha(e), \omega(e)\}$, then we have the commutative diagram

$$\begin{array}{ccc} & G_v & \\ \nearrow & & \searrow \\ G_e & \xrightarrow{\quad} & G \end{array}$$

2. The group $G = \pi_1(G, Y)$ is universal in the following sense: given any group H and a collection of compatible homomorphisms $G_v \rightarrow H$, $G_e \rightarrow H$, there exists a unique homomorphism $G \rightarrow H$ so that for all $v \in \text{vert}(Y)$, we have the commutative diagram

$$\begin{array}{ccc} & G & \\ \nearrow & & \searrow \\ G_v & \xrightarrow{\quad} & H \end{array}$$

From the above properties, it is easy to see that the fundamental group of graph of groups is unique (upto isomorphism).

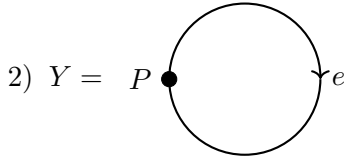
Proposition 5.3. Let (G, Y) be a graph of groups, let $P_0 \in \text{vert } Y$ and let T be a maximal tree of Y . The canonical projection $p : F(G, Y) \rightarrow \pi_1(G, Y, T)$ induces an isomorphism of $\pi_1(G, Y, P_0)$ onto $\pi_1(G, Y, T)$.

For the proof, refer to [Ser80, Chapter 5, Proposition 20].

From now we denote the graph of group simply by $\pi_1(G, Y)$ without referring to the vertex or the maximal tree.

Some examples. 1) If $Y = \overset{P}{\bullet} \xrightarrow{e} \overset{Q}{\bullet}$ is a segment, we have

$$\pi_1(G, Y, Y) = G_P *_{G_e} G_Q$$



Y is a loop. Then by the definitions, we have $F(G, Y)$ equal to $G_P * F(e, \bar{e})$ quotiented by the normal closure of $\{\bar{e} = e^{-1}, ea^e e^{-1} = a^{\bar{e}} | e \in \text{edge } Y, a \in G_e\}$.

let A and \bar{A} be the images of G_e and $G_{\bar{e}}$ in G_P respectively. Then, by the first definition of the fundamental group

$$\pi_1(G, Y, P) = G *_{P A \cong \bar{A}}.$$

5.2. Proof of Theorem 1.1. Let (G, Y) be a graph of groups.

Case 1: Suppose Y has an oriented edge $e = [v_1, v_2]$ so that removing e separates Y into two disjoint graphs Y_1 and Y_2 . We retain the vertices of e such that $v_i \in \text{vert}(Y_i)$. Let (G, Y_i) be the corresponding graphs of groups. Then let's denote

$$G_i = \pi_1(G, Y_i), i = 1, 2$$

$$G_3 = G_e$$

By the universal property of $\pi_1(G_i)$ and $\pi_1(G)$ we have, $G \cong G_1 *_{G_3} G_2$.

Case 2: Suppose Y has an oriented edge $e = [v_1, v_2]$ so that removing e does not separate Y into disjoint graphs. We retain the vertices of e to get a new graph Y' . Let (G, Y') be the corresponding graph of groups. We denote $G' = \pi_1(G, Y')$.

Then the embeddings $G_e \rightarrow G_{v_i}$, $i = 1, 2$ induces the embeddings $G_e \rightarrow G'$ with images H_1, H_2 respectively. As in *Case 1*, we get

$$G \cong G' *_{H_1 \cong H_2}.$$

Thus, the fundamental group of any graph of groups has a representation which is an amalgamated free product or a HNN extension. The statement of Theorem 1.1 now follows from Proposition 3.1 and Proposition 4.1.

REFERENCES

- [Bar97] V. G. Bardakov. On the width of verbal subgroups of some free constructions. *Algebra i Logika*, 36(5):494–517, 599, 1997.
- [BBG16] Valeriy G. Bardakov, Oleg V. Bryukhanov, and Krishnendu Gongopadhyay. Palindromic widths of nilpotent and wreath products. *Proc. Indian Acad. Sci. Math. Sci.*, 2016.
- [BG14a] Valeriy G. Bardakov and Krishnendu Gongopadhyay. On palindromic width of certain extensions and quotients of free nilpotent groups. *Internat. J. Algebra Comput.*, 24(5):553–567, 2014.
- [BG14b] Valeriy G. Bardakov and Krishnendu Gongopadhyay. Palindromic width of free nilpotent groups. *J. Algebra*, 402:379–391, 2014.
- [BG15] Valeriy G. Bardakov and Krishnendu Gongopadhyay. Palindromic width of finitely generated solvable groups. *Comm. Algebra*, 43(11):4809–4824, 2015.
- [BGS15a] Valeriy G. Bardakov, Krishnendu Gongopadhyay, and Mahender Singh. Palindromic automorphisms of free groups. *J. Algebra*, 438:260–282, 2015.
- [BGSVW15] Valery Georgievich Bardakov, Krishnendu Gongopadhyay, Mahender Singh, Andrei Vesnin, and Jie Wu. Some problems on knots, braids, and automorphism groups. *Sib. Elektron. Mat. Izv.*, 12:394–405, 2015.
- [BST05] Valery Bardakov, Vladimir Shpilrain, and Vladimir Tolstykh. On the palindromic and primitive widths of a free group. *J. Algebra*, 285(2):574–585, 2005.
- [BT06] Valery Bardakov and Vladimir Tolstykh. The palindromic width of a free product of groups. *J. Aust. Math. Soc.*, 81(2):199–208, 2006.
- [DD89] Warren Dicks and M. J. Dunwoody. *Groups acting on graphs*, volume 17 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989.
- [DK16] Cornelia Drutu and Michael Kapovich. *Lectures on Geometric Group Theory*. 2016. Available online.
- [Dob00] I. V. Dobrynina. On the width in free products with amalgamation. *Mat. Zametki*, 68(3):353–359, 2000.
- [Dob09] I. V. Dobrynina. Solution of the width problem in amalgamated free products. *Fundam. Prikl. Mat.*, 15(1):23–30, 2009.

- [Fin14] Elisabeth Fink. Conjugacy growth and width of certain branch groups. *Internat. J. Algebra Comput.*, 24(8):1213–1231, 2014.
- [Fin17] Elisabeth Fink. Palindromic width of wreath products. *J. Algebra*, 471:1–12, 2017.
- [FT15] Elisabeth Fink and Andreas Thom. Palindromic words in simple groups. *Internat. J. Algebra Comput.*, 25(3):439–444, 2015.
- [Full15] Neil J. Fullarton. A generating set for the palindromic Torelli group. *Algebr. Geom. Topol.*, 15(6):3535–3567, 2015.
- [GJ00] Henry H. Glover and Craig A. Jensen. Geometry for palindromic automorphism groups of free groups. *Comment. Math. Helv.*, 75(4):644–667, 2000.
- [GK09] Jane Gilman and Linda Keen. Discreteness criteria and the hyperbolic geometry of palindromes. *Conform. Geom. Dyn.*, 13:76–90, 2009.
- [Kot04] D. Kotschick. What is... a quasi-morphism? *Notices Amer. Math. Soc.*, 51(2):208–209, 2004.
- [LS01] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.
- [LOST10] Martin W. Liebeck, E. A. O’Brien, Aner Shalev, and Pham Huu Tiep. The Ore conjecture. *J. Eur. Math. Soc. (JEMS)*, 12(4):939–1008, 2010.
- [RS14] Tim R. Riley and Andrew W. Sale. Palindromic width of wreath products, metabelian groups, and max-n solvable groups. *Groups Complex. Cryptol.*, 6(2):121–132, 2014.
- [Seg09] Dan Segal. *Words: notes on verbal width in groups*, volume 361 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2009.
- [Ser80] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell.

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) MOHALI, KNOWLEDGE CITY,
SECTOR 81, S.A.S. NAGAR, PUNJAB 140306, INDIA

E-mail address: krishnendug@gmail.com, krishnendu@iisermohali.ac.in

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) MOHALI, KNOWLEDGE CITY,
SECTOR 81, S.A.S. NAGAR, PUNJAB 140306, INDIA

E-mail address: swathi280491@gmail.com